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ERROR ESTIMATION IN APPLIED INVERSE PROBLEMS

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Abstract - This paper presents two practical approaches to the pointwise error estimate of an approximate solution of inverse ill-posed problems. In the first approach we assume that the exact solution of an inverse problem belongs to a compact set. In this case, the problem is formulated as a well-posed according to Tikhonov and an *a priori* error estimate on the compact set is determined. The effectiveness of the approach is demonstrated by the example of an inverse problem in fluid dynamics. In the second approach we suppose that the exact solution of an inverse problem is a smooth function and apply the Tikhonov regularizing algorithm in order to find an approximate solution. We introduce a compact set of *a posteriori* constraints and estimate an error on the set of functions close to the approximate solution found in a certain sense. As an example, an inverse problem in nuclear physics is considered.

1. INTRODUCTION

The main rule that should be used before solving a practical ill-posed problem is to study all physical details concerning an unknown solution. This necessitates the inclusion of *a priori* information (*a priori* constraints) in a statement of the mathematical problem before solving. Indeed, an ill-posed problem contains undesirable features. Even if we have a regularizing algorithm, we can not estimate the proximity of an approximate solution to the exact one, [1, 2]. If additional information about the structure of a set to which the exact solution belongs is known, then in some cases one can find an error of an approximate solution, [3, 4, 5].

In this paper, we consider two applied inverse ill-posed problems. The first problem is a problem in fluid dynamics. We reconstruct axisymmetric velocity profiles of fluid or gas flow in a circular cross-section of a transport channel by using special ultrasonic measurements and solving an Abel-type integral equation of the first kind. The numerical solution of the equation implies a number of problems related to the singularity of the integrand as well as with the general ill-posedness of the integral equations of the first kind. Two approaches to the pointwise error estimate for an approximate solution of Abel-type integral equations are described in [6, 7] under the condition that the exact solution is a bounded monotonic or convex function. It is shown that one approach is more preferable than the other. In this paper, we propose a simplified version of the more accuracy method under the natural assumption on a velocity vector distribution. We suppose that the exact solution is a monotonic nonincreasing convex function bounded on a given segment. The algorithm can be applied in the case of very limited experimental data.

The second problem is an inverse problem in nuclear physics. We reconstruct photonuclear cross sections by solving the Volterra integral equation of the first kind under the condition that the exact solution of the problem is a smooth function. The Tikhonov regularizing algorithm is applied in order to find an approximate solution, [8]. We introduce a compact set of *a posteriori* constraints, using the structure of the approximate solution found. Thus, we can estimate an error at least on the set of functions close to the approximate solution found in a certain sense.

2. ERROR ESTIMATION IN THE PROBLEM IN FLUID DYNAMICS

Problems related to determining the characteristics of fluid flows in channels with a circular cross-section are of great practical interest, [9]. There are different methods of flow measurements. In this paper, we consider the technique of ultrasonic time-of-flight measurements. The magnitudes being registered in the measuring process are time-of-flight differences of ultrasonic pulses generated in opposite directions by pairs of transducers (typically mounted on the walls of a pipe and located at some distance from each other) in a number of parallel measuring planes. The flowcell design and scheme of measurements are presented in [10]. The difference between the propagation times is used to calculate the average velocity,

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which is proportional to the integral of the two-dimensional distribution of axial flow velocity along a straight line (the line of intersection of the measurement plane with the cross section of the channel). For a radially symmetric velocity distribution, a two-dimensional velocity distribution is reconstructed by solving an Abel-type integral equation of the first kind.

2.1. Problem Statement

Let ξ_j be the distance of the jth measuring plane from the channel centerline, m be the number of measurement planes (ordinarily $m = 2, 3, \ldots, 11$), R be the inner radius of the channel and L be the distance between the transducers. The flow in the channel is assumed to be vortex-free in the z-direction, i.e. $\mathbf{v} = \{0, 0, v_z(x, y)\}$. The cross-sectional distribution of the velocity vector v_z has axial symmetry and hence $v_z(x, y) = v_z(\sqrt{x^2 + y^2}) = v_z(r)$, where r is the distance from the channel centerline. The boundary condition is: $v_z(R) = 0$. For convenience, we introduce the notation $z(r) = v_z(r)$, $0 \le r \le R$. According to [10], the average velocity $u(\xi)$ in the measurement plane located at a distance $0 \le \xi < R$ from the channel centerline is related to the unknown radial distribution $z(r)$ of the true velocity by the Abel-type integral equation,

$$\frac{1}{\sqrt{R^2 - \xi^2}} \int_{\xi}^{R} \frac{r z(r) dr}{\sqrt{r^2 - \xi^2}} = u(\xi), \quad \xi \in [0, R) \tag{1}$$

It is convenient to solve (1) written in the form,

$$Az = u, \qquad z \in M \subset L^2[0, R], \quad u \in L^2[0, R] \tag{2}$$

where M is a set of *a priori* constraints. In this paper we take the set of monotonic nonincreasing convex functions bounded on $[0, R]$ as a set M ($0 \le z(r) \le C$, C is a known constant). The set M is a compact set in L^2, see, e.g. [1]. The operator $A : L^2 \to L^2$ is the linear continuous compact operator and A maps M one-to-one onto $AM \subset L^2$ (see [11]). According to Tikhonov, problem (2) is well-posed on the set M, [8]. In this case, the inverse operator A^{-1} is continuous on $AM \subset L^2$. Therefore, a small error in the function $u \in AM$ implies a small error in the solution $z \in M$. This can be used to find, together with an approximate solution, also its error.

In practice, only discrete values of $u(\xi)$ in the measurement planes are available rather than the function itself. In addition, one can find only approximate values $u_\delta(\xi_j)$, $j = \overline{1, m}$ of the average flow velocity in the measuring planes from measurement of differences in travel times $\Delta t(\xi_j)$, $j = \overline{1, m}$ by using the formula

$$u_\delta(\xi_j) = \frac{c_0^2}{2L} \Delta t(\xi_j)$$

where c_0 is the sound speed in the unperturbed medium, [10]. Thus, we switch to a finite-dimensional problem. We introduce uniform grids $\{r_i\}_1^{n+1}$, $r_i = R(i-1)/n$ and $\{\xi_j\}_1^m$, $\xi_j = R(j-1)/m$ and put $z_i = z(r_i)$, $z_{n+1} \equiv 0$, $u_j = u(\xi_j)$. We suppose that instead of the exact grid values u_j we are given approximate grid values u_j^δ and errors δ_j so that $|u_j - u_j^\delta| \le \delta_j$, $j = \overline{1, m}$. The accuracy of single-path flow velocity measurements achieved in the field are typically $\pm 1 - 2$ % at flow velocities greater than 0.3 ms^{-1} and ± 0.01 % at velocities less or equal to 0.3 ms^{-1}, [9]. Thus, the data $\{A, \{u_j^\delta\}_1^m, \{\delta_j\}_1^m\}$ are input data of the problem.

Our aim is to construct a region to which the exact solution of the problem belongs.

2.2. Solution of the Problem

Apparent from the previous subsection, we can accept the set $Z_M^\delta \equiv \{z \in M : -\delta_j \le A^j z - u_j^\delta \le \delta_j, j = \overline{1, m}\}$ as a set of approximate solutions of the problem (2), where $A^j z = (Az(r))(\xi_j)$. Each function of the set Z_M^δ satisfies equation (1) within the accuracy of the input data. The exact solution of the problem belongs to Z_M^δ, hence $Z_M^\delta \neq \emptyset$. It is impossible to construct the set Z_M^δ in practice, since any numerical method of the solution contains an error of the approximation. Therefore, we will find a set Z_M^Δ such that $Z_M^\delta \subset Z_M^\Delta$.

At first we have to construct a finite-dimensional set $\hat{Z}_M^\Delta$. It is known that the exact solution of the problem belongs to the compact set M, therefore, there exists a set $\hat{M}$ of *a priori* restrictions for a vector of grid values $\hat{z} = (z_1, \dots, z_n)$ in $\mathbf{R}^n$. The set $\hat{M}$ can be defined by $n+1$ inequalities:

$$\hat{M} = \left\{ \hat{z} \in \mathbf{R}^n : \begin{array}{l} z_1 \leq C \\ z_2 - z_1 \leq 0 \\ z_{i-1} - 2\,z_i + z_{i+1} \leq 0, \quad i = \overline{2, n-1} \\ z_{n-1} - 2\,z_n \leq 0 \end{array} \right\} \tag{3}$$

The set $\hat{M}$ is a closed convex bounded polyhedron, [1].

Let us suppose that the vector $\hat{z}$ is fixed. One can introduce a function $z_n^l(r)$ bounding the admitted region of a function $z(r) \in M$, $z(r_i) = z_i$ from below. We take the function

$$z_n^l(r) = \; z_i + \frac{z_{i+1} - z_i}{r_{i+1} - r_i}(r - r_i), \quad r \in [r_i, r_{i+1}], \quad i = \overline{1, n} \tag{4}$$

In this case, $z_n^l(r) = \inf\{z(r) : z(r) \in M, z(r_i) = z_i\}$. It is possible to introduce also a function $z_n^u(r)$ bounding the admitted region of the function $z(r) \in M$, $z(r_i) = z_i$ from above. We will use the function

$$z_n^u(r) = \left\{ \begin{array}{ll} z_i, & r \in [r_i, r_{i+1}), \quad i = \overline{1, n} \\ 0, & r = r_{n+1} \end{array} \right. \tag{5}$$

Clearly, $z_n^u(r) \geq \sup\{z(r) : z(r) \in M, z(r_i) = z_i\}$. For any function $z(r) \in M$ such that $z_n^l(r) \leq z(r) \leq z_n^u(r)$, $z_n^l(r_i) = z_i = z_n^u(r_i)$ and any $\xi_j \in [0, R)$ the following inequality is just: $A^j z_n^l \leq A^j z \leq A^j z_n^u$, where $A^j z_n^l = (A z_n^l(r))(\xi_j)$, $A^j z_n^u = (A z_n^u(r))(\xi_j)$. Consequently, the set

$$\hat{Z}_M^\Delta \equiv \left\{ \hat{z} \in \hat{M} : \begin{array}{ll} A^j z_n^l \leq u_j^\delta + \delta_j, & j = \overline{1, m} \\ A^j z_n^u \geq u_j^\delta - \delta_j, & j = \overline{1, m} \end{array} \right\} \tag{6}$$

can be taken as a finite-dimensional set of approximate solutions of the problem. The values $A^j z_n^l$ and $A^j z_n^u$ are easily calculated analytically. If we substitute (4) into (1) we obtain

$$\begin{aligned} A^j z_n^l \;\; &= \sum_{i=k}^{n} \frac{0.5}{(r_{i+1} - r_i)\sqrt{R^2 - \xi_j^2}} \left\{ \left((2\,r_{i+1} - r)\sqrt{r^2 - \xi_j^2} - \xi_j^2 \ln\left(r + \sqrt{r^2 - \xi_j^2}\right) \right) z_i \right. \\ &+ \;\; \left. \left((r - 2\,r_i)\sqrt{r^2 - \xi_j^2} + \xi_j^2 \ln\left(r + \sqrt{r^2 - \xi_j^2}\right) \right) z_{i+1} \right\} \Bigg|_{\beta_i}^{r_{i+1}} \end{aligned}$$

where $\xi_j \in [r_k, r_{k+1}), j = \overline{1, m}, k \in \overline{1, n}$; $\beta_k = \xi_j$; $\beta_i = r_i$, $i = \overline{k+1, n}$. If we substitute (5) into (1) we obtain

$$A^j z_n^u = \frac{\sqrt{r_{k+1}^2 - \xi_j^2}}{\sqrt{R^2 - \xi_j^2}} z_k + \sum_{i=k+1}^{n} \frac{\sqrt{r_{i+1}^2 - \xi_j^2} - \sqrt{r_i^2 - \xi_j^2}}{\sqrt{R^2 - \xi_j^2}} z_i$$

where $\xi_j \in [r_k, r_{k+1}), j = \overline{1, m}, k \in \overline{1, n}$.

We have obtained that both $A^j z_n^l$ and $A^j z_n^u$ are linear functions on $\hat{z}$ and the set $\hat{Z}_M^\Delta$ is the closed convex bounded polyhedron as the intersection of the polyhedron $\hat{M}$ and the convex polyhedral set. Therefore, the problem of finding an error in a solution is reduced to a linear programming problem. In our case, the linear programming problem is posed as follows. We want to find the minimum and the maximum values of each coordinate of the set $\hat{Z}_M^\Delta$. Consequently, we should minimize the objective functions

$$f(\hat{z}) = \pm z_i, \quad i \in \overline{1, n}$$

on the set (6). The problem can be easily solved, e.g. by the simplex method. Thus, we find the values $z_i^l = \inf\{z_i : \hat{z} \in \hat{Z}_M^\Delta\}$, $i = \overline{1, n}$ and $z_i^u = \inf\{-z_i : \hat{z} \in \hat{Z}_M^\Delta\}$, $i = \overline{1, n}$ such that for any $\hat{z} \in \hat{Z}_M^\Delta$: $z_i^l \leq z_i \leq z_i^u$, $i = \overline{1, n}$, i.e. we obtain the errors in n points of the segment $[0, R]$.

Using the values z_i^l, z_i^u, one can construct two functions $z^l(r)$, $z^u(r)$ bounding a region to which all infinite-dimensional approximate solutions belong from below and above, respectively. We suppose $z^l(r) = \inf\{z(r) : z(r) \in Z_M^\Delta\}$, then

$$z^l(r) = \; z_i^l + \frac{z_{i+1}^l - z_i^l}{r_{i+1} - r_i}(r - r_i), \quad r \in [r_i, r_{i+1}], \quad i = \overline{1, n}$$

where $z_{n+1}^l \equiv 0$. As a function $z^u(r)$ we take the function

$$z^u(r) = \begin{cases} z_i^u & r \in [r_i, r_{i+1}), \quad i = \overline{1,n} \\ 0 & r = r_{n+1} \end{cases}$$

Clearly, $z^u(r) \geq \sup\{z(r) : z(r) \in Z_M^\Delta\}$. Thus, $\forall z(r) \in Z_M^\Delta$ we have $z^l(r) \leq z(r) \leq z^u(r)$ and we obtain the pointwise error estimate of problem (2).

2.3. Numerical Simulation

Numerical simulations were performed in the case of $m = 2, 3, \ldots, 11$, $n = 100$, and two examples are given below. We use the known analytical expression $z(r) = a(1 - r^2)^{1/b}$ frequently in the symmetrical flow modelling as the exact flow velocity distribution. Here a, b are given numbers and we suppose that $a = 1$, $b = 1$ or $b = 6$.

Example 1. In this example, we demonstrate the effectiveness of the error estimation method in the case of $m = 2$ and $m = 11$. We use the exact input data, i.e. $u_j^\delta = u_j$, $\delta_j = 0$, $j = \overline{1,m}$. All calculations were performed using Fortran 90. The problem of the linear function $f(\hat{z})$ minimization was solved by the routine DDLPRS of the library Microsoft IMSL (the routine DDLPRS uses a revised simplex method to solve linear programming problems). Some of the results are shown in Figure 1 and Figure 2.

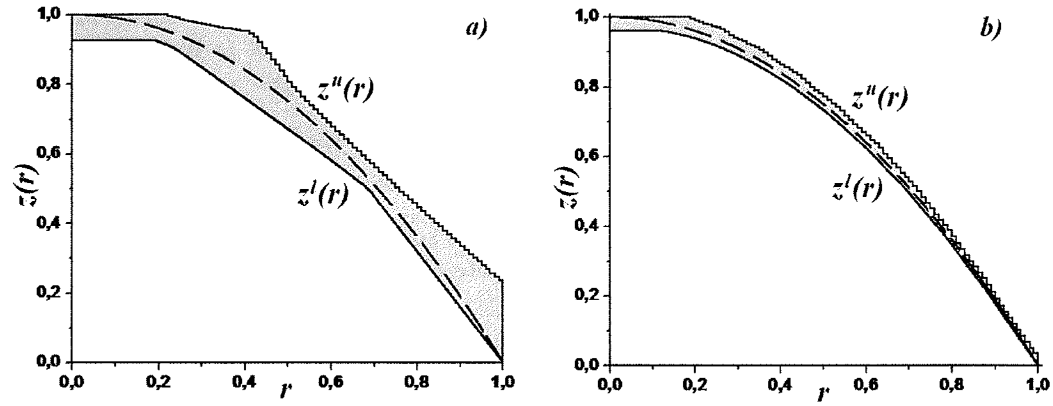

Figure 1. (a) The region of the approximate solutions (▒) in the case of $b = 1$, $m = 2$ and the exact solution (−−), (b) the region of the approximate solutions (▒) in the case of $b = 1$, $m = 11$ and the exact solution (−−).

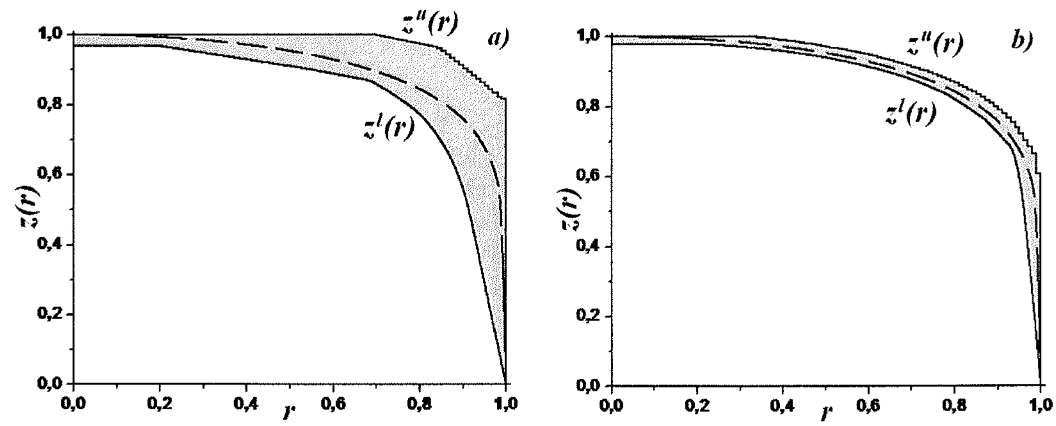

Figure 2. (a) The region of the approximate solutions (▒) in the case of $b = 6$, $m = 2$ and the exact solution (−−), (b) the region of the approximate solutions (▒) in the case of $b = 6$, $m = 11$ and the exact solution (−−).

Example 2. In this example, we demonstrate the solution of the problem in the case of inaccurate input data. In addition, the constant C is not used during the solution process, and we take $b = 6$

and $m = 11$. Solving the direct problem, we calculate the exact values of the function $u(\xi)$ on the grid $\{\xi_j\}_1^m$. Simulating the process of the appearance of random errors in the input data, we replace u_j by u_j^δ: $u_j^\delta = u_j(1 + \theta_i \varepsilon)$, where θ_i are random numbers uniformly distributed on the interval $(-1, 1)$. If we take $\varepsilon = 0.01q$, then the relative error in the input data u_j^δ in comparison with u_j will not exceed $q\%$ and we take $q = 1$. The results are shown in Figure 3.

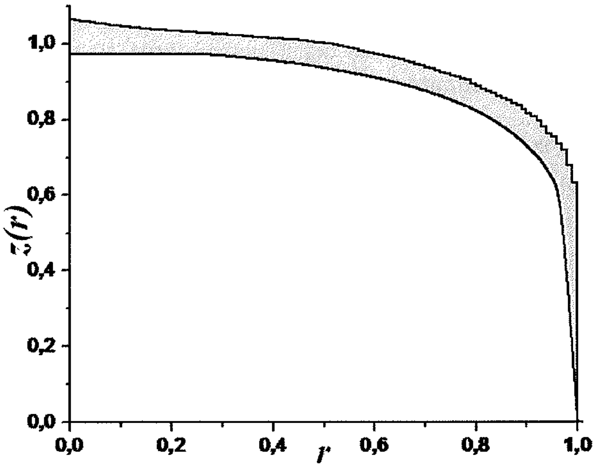

Figure 3. The region to which all approximate velocity profiles and the exact velocity profile belong.

Remark. We have constructed the region to which all approximate solutions belong. If one needs to choose a fixed approximate solution, any vector $\hat{z} \in \hat{Z}_M^\Delta$ may be taken in principle. We recommend to take an element $\hat{z}_\Delta \in \hat{Z}_M^\Delta$ such that the quadratic function $\varphi(\hat{z})$:

$$\varphi(\hat{z}) = \sum_{j=1}^{m} (A^j z_n^l - u_j^\delta)^2$$

takes the minimum value, i.e. $\varphi(\hat{z}_\Delta) = \inf\{\varphi(\hat{z}) : \hat{z} \in \hat{Z}_M^\Delta\}$. The minimization problem, subject to the linear inequality constraints (3) and (6), can be solved very fast, e.g. by the conjugate gradient projection method, [1].

3. ERROR ESTIMATION IN THE PROBLEM OF NUCLEAR PHYSICS

There are numerous inverse ill-posed problems where it is impossible to introduce a compact set of *a priori* constraints, but it is known that the exact solution is a smooth function and has a finite number of extrema and inflection points. In this case, one can use the Tikhonov regularizing algorithm in order to find an approximate solution, [1, 8]. Then, we suggest to use a compact set of *a posteriori* constraints and estimate an error in the approximate solution at least on this set. The approach is considered by an example of an inverse problem in nuclear physics.

3.1. Problem Statement

Let a sample and a suitable monitor be simultaneously irradiated by a bremsstrahlung beam of maximum energy T. If $W(T, E_\gamma)$ is the number of photons of energy E_γ (per unit range of E_γ) which enter the sample per unit of monitor response, $\sigma(E_\gamma)$ is the desired photo cross section in cm^2 per nucleus, and μ is the number of nuclei of the appropriate type per cm^2 of sample, then the number of reactions which occur per unit of monitor response, $Y(T)$, is given by the following Volterra integral equation of the first kind, [12]:

$$\mu \int_{E_{min}}^{T} W(T, E_\gamma)\sigma(E_\gamma) = Y(T), \quad T \in [E_{min}, E_{max}] \tag{7}$$

where E_{min} is the threshold of a reaction, $W(T, E_\gamma)$ is the Schiff integrated-over-angles bremsstrahlung spectrum, [13]. The measurements are repeated for a series of values of T and a series of points on the bremsstrahlung yield curve, $Y(T)$, are obtained. We may conclude from a priori considerations that the exact photonuclear cross section corresponding to the exact $Y(T)$ is a smooth function on $[E_{min}, E_{max}]$. We suppose that instead of the exact function $Y(T)$ we have an approximate function $Y_\delta(T)$ such that $\|Y(T) - Y_\delta(T)\|_{L^2} \leq \delta$ and the error $\delta > 0$ is known. In this case, the problem (7) can be written as follows:

$$A\sigma = Y, \qquad \sigma \in D \subset W_2^1[E_{min}, E_{max}], \quad Y \in L^2[E_{min}, E_{max}] \tag{8}$$

where $D = \{\sigma \in W_2^1 : 0 \leq \sigma \leq C\}$, and C is a known constant.

Our aim is to find an approximate solution which converges, as $\delta \to 0$, to the exact solution in the norm of the space $W_2^1[E_{min}, E_{max}]$ and to construct an area to which all solutions having the same structure and close to the approximate solution found belong.

3.2. Solution of the Problem

We use the standard scheme for constructing a regularizing algorithm, [1, 8]. We introduce the Tikhonov functional $M^\alpha[\sigma]$:

$$M^\alpha[\sigma] = \|A\sigma - Y_\delta\|_{L^2}^2 + \alpha\|\sigma\|_{W_2^1}^2 =$$

$$= \int_{E_{min}}^{E_{max}} \left[\mu \int_{E_{min}}^{T} W(T, E_\gamma)\sigma(E_\gamma)dE_\gamma - Y_\delta(T) \right]^2 dT + \alpha \int_{E_{min}}^{E_{max}} \{\sigma^2(E_\gamma) + [\sigma'(E_\gamma)]^2\} \, dE_\gamma$$

where α is a regularization parameter. The parameter α is chosen in accordance with the discrepancy principle, [1, 8]. The extremal σ_δ^α of the functional $M^\alpha[\sigma]$, i.e. an element minimizing $M^\alpha[\sigma]$ on D, is taken as an approximate solution of equation (8). Numerical algorithms for the practical solving of this type of problem are described in detail in [1] as well as numerous computational procedures for their realization.

Let the approximate solution σ_δ^α we found using the Tikhonov regularizing method. Suppose that the solution has a limited number of extrema and inflection points. We divide the segment $[E_{min}, E_{max}]$ into s parts $[E_i, E_{i+1}]$, $i = \overline{1, s}$, so that $\sigma_\delta^\alpha(E_\gamma)$, $E_\gamma \in [E_i, E_{i+1}]$, $i \in \overline{1, s}$ is a monotonic, convex or monotonic and convex function. In this case, one can introduce a compact set M of *a posteriori* constraints and find an error estimate at least on this set by using a method similar to the algorithm described in the previous section.

3.3. Numerical Example

In this section we will demonstrate the effectiveness of the approach by a model example. Let the exact cross section be defined as follows:

$$\sigma^{mod}(E_\gamma) = \sum_{i=1}^{4} \frac{a_i \Gamma_i^2}{(E_\gamma - b_i)^2 + \Gamma_i^2}, \qquad E_\gamma \in [5, 60]$$

where a_i is a height of the ith resonance, b_i is a location of the ith resonance and Γ_i is a width of the ith resonance. We suppose $a = (15, 25, 20, 7)$, $b = (15, 25, 35, 45)$, $\Gamma = (2, 4, 5, 2)$. Solving the direct problem, we calculate the exact grid values $\{Y_j\}_1^{20}$ of the yield curve $Y(T)$. These values are perturbed by errors in such a way that the following inequalities are just: $|Y_j - Y_j^\delta| \leq 0.01Y_j$, $j = \overline{1, 20}$. The solution is reconstructed by using the Tikhonov regularizing algorithm in $n = 100$ points. The results of the reconstruction and the pointwise error estimate are shown in Figure 4.

It is to be noted that the structure of the region to which a given set of approximate solutions belong is directly related to *a posteriori* constraints utilized after the Tikhonov regularization. In this example, we have used the extrema and the inflection points location of the found solution.

4. CONCLUSION

We have theoretically investigated the possibility of the error estimation in two different inverse problems and concluded that the methods considered above can be applied for the data processing. In the inverse problem in fluid dynamics we assumed that the exact solution is a monotonic nonincreasing convex function bounded on the segment $[0, R]$ and determined an *a priori* error estimate. In the inverse problem in nuclear physics we introduced a compact set of *a posteriori* constraints and estimated an error at least on the set of functions close to the approximate solution found in a certain sense.

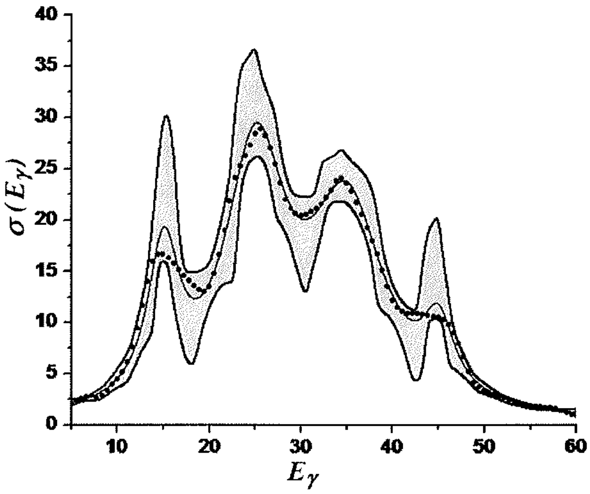

Figure 4. The exact solution (—), the approximate solution found using the Tikhonov regularizing algorithm (···), the region to which all approximate solutions close to the solution found belong (▒).

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